Review of July 2011 draft of Texas Essential Knowledge and Skills – Mathematics

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Some parts of these standards have been significantly improved from the previous draft, but there is still work to be done.

Language is important. It can be used to suggest to teachers, parents and students what is expected. Therefore it is important to be careful in the use of words. In the introduction there is the following sentence: "By embedding statistics, probability, and finance, while focusing on fluency and deep understanding, Texas will lead the way in mathematics education and prepare all Texas students for challenges they will face in the 21st century."

Let us consider "understanding". Understanding comes at different levels. One starts to understand something, but initial understanding is usually fragile in the sense that when this is needed in a slightly different setting, often the connection is not made, and if a somewhat subtle question is asked, the understanding seems to evaporate. Then there is solid understanding, which can be used in somewhat different settings and it does not disappear when serious questions are asked. Deep understanding, goes well beyond solid understanding. It can be said to exist when the understanding can be used to project into the further developments which go well beyond what led to solid understanding. It is unrealistic to expect that this can be taught in school. There are many instances where even solid understanding is not supported in the current draft.

Here is another example where the use of language causes confusion in this draft. Standard 7A02 is:

"<u>illustrate</u> and explain the relationship between the volume of a rectangular prism and a rectangular pyramid having both congruent bases and heights, such as, the volume of a pyramid is 1/3 the volume of the prism that has the same base area and height."

Compare this with Standard 8A04:

"illustrate and explain the relationship between the formula for the volume of a sphere as it relates to the volume of a cone whose base radius and height are equal to each other and are congruent to the radius of the sphere."

Since the same words introduce both standards, this suggests that the writers think the same level of knowledge is expected in both cases. However, that should not be the case. It is easy to give a mathematical motivation for the factor of 1/3 in the first standard. Take a cube, and from the center point draw lines to the vertices. This cuts the cube into six congruent square pyramids so the volume of each is 1/6 times the volume of the cube. The height of each pyramid is ½ of the length of an edge of the cube, so there is a factor of 1/3 for the volume of this pyramid in comparison to the volume of a prism with the same base and height. This is not a proof of the formula, but it can be made into a proof in high school. I have used the argument with a cube with elementary school children after having discussed the

area of a triangle with them, and they have no trouble understanding it. There is no such simple argument for the connection of the volume of a sphere and of a cylinder. The use of the same words in these two standards either asks for too little in the first case or too much in the second. My advice is to include an illustration of what can be done in the case of a square pyramid, since both textbook publishers and teachers need information like this. Then, in high school, include standards which say that students taking Precalculus should be able to derive the formulas for the volume of a pyramid, of a cone, and of a sphere. I still remember this from high school, and it was very useful as a way to help me understand integration when I learned it in calculus. The effect of scaling on area and volume seems to be missing in this draft. An informal version of this should be given in middle school, say grade 8, with a somewhat more detailed version in high school geometry. The scaling for area occurs in grade 7 in the following way. When the constant π is introduced in 7P06, a student is expected to represent π as the ratio of the circumference of a circle to its diameter and as the area of a circle to the square of its radius. This is not what should be written. The ratio of the circumference of a circle to its diameter is a constant and the ratio of the area of a circle to the square of its radius is a constant. One of these constants is called π, and an informal argument then needs to be given to show that the other constant has the same value. The point, which is important, is that one should not give two different definitions for something without showing they are equivalent or that one implies the other. Without this knowledge, understanding of definitions is not solid. This does not come all at once, but when this basic idea is ignored, students will understandably get incorrect views of mathematics and of general reasoning. My own preference is to call the equality of these two constants a minor miracle, but this might not be appropriate in a set of standards.

A major point deals with what should be taught in class so that at least some students will get a deeper understanding than seems to be suggested. In my previous report I mentioned the case of a textbook for Algebra 2 which had no derivation of the sum of a geometric series, finite or infinite, and being told that the publisher includes what is in state standards. What was really meant by that answer is that if something is not explicitly stated in the standards, there is a fair chance it will not be in a textbook. There are many instances where this draft says students should use something, such as PM04: "use the Law of Sines in mathematical and real world problems", but nowhere does it seem to require a derivation of this simple result. Texas may think that not all students need to know how to derive the Law of Sines, but students who will take mathematics in college should know how to derive it. There is little excuse for not requiring this in Precalculus. One cannot claim to have solid understanding of trigonometry without being able to derive this result, and some others which are just mentioned as something students should be able to use. Another example is the addition formula for *sin* (*A+B*) and also the related one for *cos* (*A+B*). All that is written about these vital formulas is in PG14 where students are to use some trigonometric identities including these two, but nothing is written about there being a derivation of them. There is a comment that this standard was moved from Geometric Reasoning to Algebraic Reasoning to better delineate the focal area. Uses of these formulas occur both in algebra and geometry settings, but the derivations of them in school mathematics are geometric.

One possibility is to write a comment that textbooks should include enough information so that a diligent student can learn how to derive various facts, and put a star next to a standard where this is

appropriate. A better one is to say that these results are important and students should know how to derive them and to use them.

There are some standards which are a significant improvement over what we get in many current textbooks. For example, 8P04 has a description of the slope of a line, and the remark that similar triangles are used to show that slope is well defined for a line is given. Few textbooks include this. This should be followed up with the corresponding fact that the graph of a linear equation in the plane is a straight line and when a line if graphed in a coordinate plane its points satisfy a linear equation. Currently the best that many students see is a picture that makes these facts look like they are true, but no follow-up argument that they are true. Mathematics education can start with something being true because it looks like it is, but it cannot stop there and be considered mathematics.

Kindergarten

I think the introduction to the Texas Essential Knowledge and Skills for Mathematics should come at the start before individual grades. A shorter version could be given for each grade, dealing with specific topics to focus on.

Grade 1

New Standard after 1G03. Relative to sharing, equal shares is a mathematical way of writing this, fair shares is not. There are many situations in which a fair share is not an equal one.

The last standard in 1G I have no idea how to do this. How does one justify that fractional parts are Halves by constructing non-examples? I do not see any reason for introducing fractions in Grade 1. Work a bit more on whole numbers so as not to fall too far behind. I would drop both of these standards for Grade 1 and put in rewritten standards in Grade 2. This standard is repeated after 2N08. It should be changed there or just dropped. As a possible substitute, try: "give examples of fractional parts which are halves and fourths", but not in grade 1.

Grade 2

There are too many different items in the new standard after 2N01. Skip counting and place value are not taught by subitizing. Skip counting and place value are both important and they should be included. Subitizing is more of a lower grade skill and it is not needed here. What should be built up by the end of grade 2 is knowing the basic addition facts, having them in long term memory and the ability to use them in addition and subtraction problems.

Second new standard in Expressions, Equations and Relationships. Why introduce even and odd numbers this early? The reason given is a comment by Rath. However, his posted comments have nothing on this. In Japan and Singapore, even and odd numbers are only introduced after division is done to the point where students can do some division.

Grade 3

3N01 Why not start with "compose and decompose numbers to 10,000 as a sum…". Yes, one can make a case that using "value of a numeral" is more precise than "numbers", but it is pedantic. At this age and for large numbers, pictures or tally marks are better than objects, so I would drop "objects".

3N20 Two members of the State School Board pushed for knowing multiplication facts through 12 by 12. Let me try to explain why I think this is a bad idea, and what I propose as a substitute. The multiplication table up to 10 by 10 is the basis for all multiplication, and with a knowledge of place value and how multiplication work one can easily find the product of 12 times 7 by reasoning that 10 times 7 is 70, 2 times 7 is 14, and 70 + 14 = 84. Why should one have to memorize this fact when it is a very simple mental calculation? It is not fundamental as the multiplication facts up to 10 are, and we want students to know what is fundamental and what is not. I would add a little more which students should learn. This is the squares up to say 15 squared. 11 squared is 121, 12 squared is 144. Let me write the squares from 11 squared to 13 squared as follows:

$11^2 = 100 + 20 + 1 = 10^2 + 2 \times 10 \times 1 + 1^2$

$12^2 = 100 + 40 + 4 = 10^2 + 2 \times 10 \times 2 + 2^2$

$13^2 = 100 + 60 + 9 = 10^2 + 2 \times 10 \times 3 + 3^2$

Each of these has the form $(a + b)^2 = a^2 + 2ab + b^2$ with a = 10 and b = 1, 2, 3. It is far too early to introduce the general form, but having these examples to refer to when it is introduced will be helpful. Twelve squared = 144 is the only one of the 12s in the 12 by 12 table which comes up often and cannot be done directly as 12 time 7 was done, and for those who want to 12 tables, the inclusion of a few squares contains the one multiple of 12 which is likely to come up and might cause students to have to work it out with pencil and paper rather than mentally.

3G04 Why include decomposing into two rectangles? A figure like H made with rectangles should be included in what students should be able to do. I would just drop the word "two".

3M08 and 09 Are fractions larger than 1 included here?

Grade 4

4N17 Students should learn that when adding or subtracting fractions which have the same denominators, it is done just as addition and subtraction of whole numbers. This should be written somewhere.

4N25 Students are expected to know the multiplication table up to 10 by 10, so what role does partial quotients play in division by a one digit number? This is what used to be called short division, and only the answer used to be written since it was mental mathematics. We are unlikely to go back to that, but the wrong message is sent to teachers about what students should know when partial quotients are listed in the standards for something like this. My advice is to stop after the word "divisor" and leave the methods of how this is done to teachers.

4A01 It is an interesting exercise to ask late elementary and middle school students what a variable is. They cannot tell you. I would leave it out of this standard.

4A03 Here, implicitly the side lengths are positive integers. There should be a fifth or sixth grade standard where the lengths are allowed to be fractions.

4G02 Why are the names "acute" and "obtuse" essential for future work? They can be useful, but we could get along without them. Having a name for a triangle with a 90 degree angle is essential. What is also essential is for students to know what a right angle is, and being able to fold one from a blob of paper is a way of helping students learn this. They learn that it is half of a straight angle, and can be used to determine if another angle is less than or greater than a right angle. I do not object to including the words acute and obtuse, but would like more stress put on right angles and right triangles.

Grade 5

5N06 The word *uncommon* jars. What would be better would to use *equal* rather than *common* for the case when one wants the denominators to be the same, and *possibly unequal* rather than *uncommon*. A replacement without using *equal* would be *possibly different*.

5N06 and earlier 4N17 The main function of this standard is addition and subtraction of fractions. Decimals are just another way of writing certain fractions. The last sentence makes little sense. "This includes fractions as decimals with common denominators of tenths or hundredths, such as 1/5 + 0.3." The standard deals with fractions with different denominators, not common denominators, and the example of 1/5 + 0.3 does not have common denominators. I think it is a mistake to include this last sentence. Students should know that 0.3 is just another way of writing 3/10, and that should be included in a standard which says students should know this not just of 0.3 but in general, here just to hundredths. How often have any of you had to add a fraction and a decimal? Of course students should be able to do this, but putting this example in the standards encourages teachers to spend time on such problems and test developers to include them. That is not a good way to get better tests or a better education for our students.

5A01 Please go back to using the word "letter" rather than "variable". It is a letter, and later students should learn that a variable is a letter which stands for a number or numbers in a set of numbers, and eventually they learn that it can be more general than that. However, in grade 5, knowing that a letter can stand for a number or a set of numbers is fine.

I.M. Gelfand was one of the great mathematicians in the second half of the 20th century. He and a colleague, A. Shen, wrote a book called *Algebra*, which is an introduction to algebra. On pages 15-17 there is a section titled *Letters in algebra*. There is a nice treatment of some of the ways letters are used in algebra, including a nice treatment of a word problem where the problem is first solved using words, then solved using shorter versions of the words, and then using letters to stand for the words. The book is 149 pages long, covers much of a first algebra course with some arithmetic at the start and a substantial amount of second year algebra in the last part of the book. The word *variable* is never used. I

will bring a copy next month in case you do not have access to it. Since the word *variable* will be used in US textbooks, it has to be introduced eventually, but not in fourth or fifth grade.

Grade 6

Last sentence in the Introduction: I do not understand or believe the statement that "emphasis on algebra readiness skills necessitates the implementation of graphing technology." Almost a year and a half ago I was in Hong Kong and sat in some classes. One was doing sixth grade material and the students in pairs played a game involving solving linear equations. A 5 by 5 grid was given with numbers in each square, and 25 linear equations were also given. The students played rock, paper, scissors to see who would start first, and the first one solved one of the equations and marked off with a circle the number which was the solution. Then the other player solved an equation and marked off the number on the grid with a cross. Then rock, paper, scissors was done again to see who would be first in the new round, and both solved other equations and marked out the corresponding numbers on the grid. The aim was to get 5 in a row. To do well one does not just solve any equation, one looks for one which will have a solution which helps getting 5 in a row or helps block the other one. The equations were not very messy, but could involve equations like $3x-5=x+29$ as well as simpler ones. The class had about 40 students and they all started to play right away and had no problems in solving the equations.

New standard after 6N-4 This standard does not get at what division is, it is an operation which undoes multiplication, nor does it get at how division is done. How is one to generate an area model for $2.5/0.5$ without first knowing what number $2.5/0.5$ is? The standard should be something like "know that division of fractions and decimals can be represented in terms of multiplication, just as division of counting numbers comes from multiplication. For example $a/b = x$ means that $bx = a$. For example, $(7/4)/(1/2)=7/2$ since $(1/2)x(7/2)=7/4$.

6N06 Why use the words "proper" and "improper" when dealing with fractions. Fractions are numbers of the form a/b where at this stage a is a whole number and b is a positive whole number. In the statement of the standard replace proper and improper with *recall a fraction can be less than one or greater than one*. This may lead to a change earlier if initially in the early work with fractions all fractions were assumed to be between 0 and 1.

6N07 This is a very surprising standard. Integers had not been mentioned to this point and now instead of a definition of description of them and how they relate to the number line, students are expected to have this as background knowledge to do operations with them. There should be an earlier standard about student knowing how to define negative integers and locate them on a number line (which has now been extended to the left of zero).

6N08 This disrupts what should be continuity from 6N07 to 6N09 so move it before the introduction of negative numbers.

6N09 This is the shortest standard for something which takes a few weeks to teach which I have seen. Should students have some idea why $(-2)x(-3)=6$? I hope so, but this standard says nothing about that.

One possible solution is just do addition and subtraction of integers in Grade 6, and do multiplication and division in Grade 7

On page 1 of Grade 6, understanding and using percents is a supporting topic. Where is there something about what percents are and why one uses them?

6A08 Move this before 6A06.

Standard before 6A10 I do not know what is expected here. For the sum of the angles of a triangle, should students know this because they measure angles, because they tear off the corners of triangles and put the angles together, or from some mathematical reasoning? What is expected about the connection between sides and angles of a triangle? In Singapore, they have the equality of base angles for an isosceles triangle shown by paper folding, and then this is used to solve very interesting multistep problems. Is this expected, or is this something qualitative like a larger angle has a larger side opposite it?

6A10 I pointed out that the illustrated argument was not sufficient, and this led to dropping the argument. That will insure that a full argument is not given in books, since it is likely never done completely in middle school books and only infrequently in high school books. Put the picture back but add a comment that this does not handle the general case but will suffice until high school when the general case will be done. The do it completely in high school.

Grade 7

First standard which was just added. Where have sets, subsets, unions, and intersections been mentioned before? We spent a year in the Netherlands when our children were in elementary school. Our son was in fifth grade that year. When we got back and school started, he said the following:

"Dad, when we were in first grade we studied sets, then in second, third and four grades we studied sets. We did not study them last year, but once again the year starts with sets. Do we have to do this again?" I said no, while sets are important, there is no reason to make their use explicit for a while beyond sixth grade. If you want to introduce this vocabulary now, you better provide illustrations for why it is useful now, not three years from now. Vocabulary without any serious use of it tends to distort mathematics.

7N01 A little more on why the product of two negative numbers is positive would be useful. Teachers need some help on this. $0 = (-1) \times 0 = (-1) \times (1 + (-1)) = (-1) \times 1 + (-1) \times (-1) = -1 + (-1) \times (-1)$. But $0 = (-1) + 1$ so $(-1) \times (-1) = 1$ might help after a few words have been added.

7P05 DROP THE WORDS INVARIANT AND COVARIANT. Last time I wrote this in the usual form, but nothing was changed, so this time the change is listed in caps. While at the State Board hearing, I asked one of the speakers what invariant and covariant meant in this setting. The response was that invariant means does not change and covariant means there is a change. That was not what was meant. The words are not necessary and will only cause confusion. Drop them. Put a period after "figures" and then include the part from (if a, … to b/b').

7P06 See page 2 above for my comments on this standard. I included a similar statement in the first report, but no change was made. There needs to be a change.

7A02 and 03 Again, what is meant by these standards. If all that is meant is that students should know there is a factor of 1/3 in the volume of a pyramid compared to a prism, why have two standards to say the same thing?

Grade 8

8N01 I would drop cube roots from this standard. If they are to be included, write something like: For cube root, for any number beween 1 and 1000, locate the cube root of it between two consecutive positive integers. For example, cube root of 500 is between 7 and 8 since 7^3 = 343 and 8^3 = 512.

8N05 The average distance beween the earth and the moon is a reasonable thing to deal with, but the average distance from the earth to Mars is not. How about replacing this with the average distance from the earth to the sun and the average distance from Mars to the sun, or the earth to the sun and the earth to the moon?

All of the standards which start with "illustrate and explain" should be rewritten so it is clear what is being asked.

8A06 REWRITE Students should be able to state, use and give a proof of the Pythagorean theorem, and be able to state and use its converse. The converse is subtle and being able to state it correctly and use it is sufficient. Few college calculus students know how to state the Pythagorean theorem. They just write $c^2 = a^2 + b^2$ without saying what the letters represent. This is an illustration of how far geometry has fallen apart.

8G01 I have no idea what this means.

8G03 Also, limit rotations to rotations with center at (0,0).

Algebra I

A1Q06.5 Knowing the roots is not sufficient even with a graph since another point has to be labeled. Since systems of two linear equations have been introduced and solved, one could add a second class of graphs where students should be able to find the equation, when one of the points has $x = 0$ as its first coordinate. For example, given a graph with the points (2, 0), (0,-4) and (-1, 0) on the graph. Find a, b, c when $y = ax^2 + bx + c$ is the equation of the graph of the curve.

Algebra II

Summing an arithmetic series and a finite geometric series should be in Algebra II, and students should know how to derive each of these and be able to use them. The finite geometric series has some important applications to financial mathematics, which should be included.

A2Q02 Why include focal width? This just adds to a long list something which is not needed. What we want is for students to be able to start from scratch and find the equation of a parabola when given the focus and directrix as a step in learning how to find equations of curves defined by various geometric conditions.

A2Q08 Equations have solutions or roots, functions have zeros.

A2C What type of cubic equation can students be expected to find? Clearly given the volume of a cube one can find the length of each edge by taking a cube root. What else do you have in mind? I would much rather have students learn how to solve a system of two equations, one linear and the other quadratic. For example, solving for the points of intersection of a line through (-1,0) with slope *2* and the unit circle with center at (0,0) can be done. It leads to a quadratic equation which can be solved by factoring or using the quadratic formula, and then generalized to a line with slope *t*. One gets infinitely many Pythagorean triples from this by taking different rational values of *t*.

Geometry

GA01 This should first be done for a segment on the real line, and then taken to the plane. The second step uses similarity, so this needs to be done earlier or else the gap needs to be mentioned and fixed later in the course.

GA02 Before dealing with parallel and perpendicular lines via coordinates, it would be a good idea to show why the equation of a line is a linear equation and why the graph of a linear equation is a line. Then in GA02, are students to be given the conditions of parallel and perpendicular, as happens in many textbooks, or will there be reasons why these conditions work?

GA03 Since a distinction has been made between "prove" and "verify", why was "prove" replaced here?

GG08 This is now in a section dealing with coordinate geometry so when you ask for rotations you are doing this in the context of coordinates, and so need formulas for rotations. You cannot do this yet since some of the needed machinery has not been introduced yet.

GG09 Same comment as above in GG08.

GG03 Drop the *s* in counterexamples.

GG11 Are the triangle congruence conditions to be proven before they are used or are they just to be stated?

GG15 Replace "and" with "or".

GS04 What does this mean? I do not understand what is meant by "generalize". Does it mean to draw a line from a vertex perpendicular to the opposite side and see what relations can be found? That is a very useful thing to do, and it gives the area of a triangle, the angle sum of a triangle, the law of sines, the law of cosines, and the addition formula for sin(*A+B*) in the case when the angles are acute. Some of these become trivial when the triangle is a right triangle but others generalize very important results such as

9

the law of cosines generalizing the Pythagorean theorem, but really being a consequence of it. What is wanted here?

GM05 and 06 The formulas mentioned here were either motivated or stated in lower grades. Is more going to be done with them here or not? If so, say so. If not, when will arguments be given to show at least informally that these formulas are true?

GC01 Are these theorems going to be derived or just statements which are true because they look like they or true or someone told me they are true.

Mathematical Models

MMAA06 I agree that this should be done with technology, but am old fashioned enough to think that students learning this should also learn how to sum a finite geometric series.

MMAD02 In school mathematics, measures of central tendency have been taken to be the mean, the median, and the mode. The mode is not a measure of the center, and the other two are measures of a center but not of central tendency. What you want is "summary statistics" which was used earlier.

MMAD04 When will the binomial theorem be done so that students can make sense of the binomial distribution and the same for an infinite geometric series so the geometric distribution is more than just a name for something pulled out of the air.

Pre-Calculus

PG13 How students expected to do this? There is a very nice article on this in an NCTM Yearbook in 1987. R. Teukolsky, Conic sections: An exciting enrichment topic, in *Learning Geometry, K-12*, ed. M.M. Lindquist, NCTM, Reston, VA, 1987, 155-174. The author was a high school teacher and her push to learn this material, use it with bright students, and write the paper to share with others, came from a student who asked why the claim about conics cut by a plane was true. What do you expect students to do, and how are they to understand what a plane is in three dimensional coordinates if this is one way a teacher tries to explain why this works? If all you want is to have students recite something they have been told which had not reasoning behind it, say so and tone the rhetoric about "deep understanding".

PM01 The phrase omitted was not instructional material. The word "evaluate" is not what you want. You want to "extend" the definition of trigonometric functions to angles larger than a right angle. Here is a rewrite of the last part: "and use this to extend the definitions of the trigonometric functions to all real numbers so they can be used to study problems where periodic behavior occurs."

PM04 and 05 For both of these, there needs to be a derivation and students should know how to derive at least the law of sines. In East Asia, the law of sines has a fourth part, the connection with the radius of the circle which contains the vertices of the triangle. The Common Core missed this. You could do them one better by including it, and maybe getting it in some textbooks.

Similar comments apply to the sum of an arithmetic series, a finite geometric series, and the addition formulas for sine and cosine. Proofs have to be included and many students should know how to derive these formulas. The infinite geometric series is important enough so it should also be included, which forces a little bit on limits to be included. Limits should be treated somewhat seriously in calculus, but there is a better chance of this happening if students have seen some limits in a non-calculus setting. One example is the sum of a convergent geometric series, and another was suggested much earlier, filling in the gaps left when the volume formulas for a pyramid, cone and sphere were stated with only a brief motivation for a special pyramid in what I wrote. This is marvelous Precalculus material and should be included.

PA18 and the split standard following it: Why are these included? Yes, such inequalities play a little role in calculus, but they are just technical with no new ideas. Students need skill, but this is not a particularly good place to develop useful skills. One or two simple nonlinear examples, such as x^2 and $1/x$, should be sufficient.

PG14 Comments on the addition formula part of this were given near the start of this review.

Advanced Quantitative Reasoning

AQRN07 How are combinatorial results or reasoning going to be developed so they can be used on what seems a very hard problem?

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Nothing has been done about having some sections of courses cover topics which might not be essential for all students, but are essential for students who want to study mathematics, science, economics, and similar subjects when they go on to college. The Precalculus course as described is not adequate for this, nor is the geometry course as it is likely to play out. Texas has done a reasonably good job in pulling up the bottom and probably the middle groups of students. I am basing this on the Texas NAEP results when split by race. What has seemingly not happened is bringing up the upper part, or keeping the edge formed in the earlier grades through high school.

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