Review of the Commissioner's Draft of the Texas Mathematics Standards

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There are four parts to this review.

1. In the first I follow the suggested "guiding questions" and discuss those of my observations that are relevant to them.

2. In the next section I discuss and illustrate a number of the kinds of issues that are present in the document. It is worth noting that the illustrations there cover only a small number of the problems with the document. It would be an overwhelming job to give a complete discussion here.

3. In the third section I discuss an issue that occurs over 200 times in the document - the failure to distinguish between "real-world problems" and mathematical problems. Since there are very few discussions of this topic available, I explain in detail what the differences are between these two topics, and show that Polya's basic four step procedure for solving problems in mathematics has to be extended by two further steps to even talk about resolving real-world problems.

4. In the final section I sketch the key steps needed to give students a strong understanding of base 10 place-value, and place-value notation. This is another key area where many teachers - and consequently their students - have difficulties.

Summary. Overall, I found the document to be filled with problems of all kinds including improper use of mathematical term, misprints, failure to define any of the terms - especially the more unusual ones, disorganized lists of standards, and a few mathematical errors. Moreover, many of the standards appear as much as two years later than they appear in the top standards and the standards of the high achieving countries. In particular this document is not benchmarked at the level of the expectations in the high achieving countries.

Some crucial topics, most notably the standard algorithms for arithmetic, are missing. Indeed, the only algorithms that are explicitly mentioned here - such as partial quotients for division - are very inefficient, and do not give students any feel for algorithms and their construction. But developing this feel is one of the major things students need to learn if they are to use mathematics effectively.

Additionally, the approach to geometry does not seem to be well thought out. The authors appear to have tried to emulate the approach suggested in Core Standards, but

those methods are extremely difficult to teach, and very few high school math teachers have the background to use them without extensive help. This made it necessary for Core Standards to develop the material very carefully, with detailed definitions of key concepts as well as extensive discussions of what the definitions mean and how they should be used. This material is completely absent in the current document, and this is a major problem.

There is one strong positive in these draft standards. Unlike too many sets of standards, including Core Standards, that are encumbered by too many pedagogy standards, these standards strictly avoid any standards of this kind.

If there are questions or further details that I can provide, please do not hesitate to contact me.

Guiding Questions

1. Is a complete and logical development of mathematics concepts followed for each grade level or course? What recommendations do you have for improvement?

> *Too many of the standards are almost incoherent. There is no obvious overview of the subject and how one topic develops from another. Without this overview, the standards become disorganized lists, nothing more. For example, consider the sequence of second grade topics 2N06, 2N07, 2N08 which discuss one of the most difficult second (or even third) grade topics, representations of fractions:*
>
> 2N06 decompose a strip diagram or regular polygon into equal parts using objects and pictorial representations.
>
> 2N07 identify and name one part of an equipartitioned whole as a fraction 1/b (where b is a non-zero whole number) using strip diagrams and area models that include regular polygons.
>
> 2N08 determine the missing value in a number statement where two fractions with like denominators form one whole, represented with a strip diagram. (e.g., 2/7+■ = 7/7. A strip diagram is separated into 7 equal parts. Two of the parts are shaded blue, and the remaining parts are shaded a second color.)
>
> First, immediately preceding this is a low level whole number standard that only discusses notation:
>
> *"represent the comparison of two numbers to 1000 using the symbols $<$, $=$, $>$."*
>
> And following is 2N09, which, aside from an egregious misprint, is an equally low level whole number standard:
>
> *"determine the number that is and 10 or 100 more or less than a given number between 100 and 900."*

One can find the same sorts of illogical orderings of topics at virtually all grade levels. The standards cannot be ordered this poorly and maintain any sense of coherence.

2. Have the correct vocabulary and terminology been used? Where can changes be made for accuracy and/or clarity?

> *For the most part, but not always, correct vocabulary is used. But the sentences that use this terminology are often entirely incoherent. For example, consider the seventh grade geometry standard 7P05*
>
> 7P05 generalize the critical attributes of similarity, including invariant and covariant relationships. (If a, a' and b, b' are side lengths of two pairs of corresponding

sides, then $a/a' = b/b'$ and $a/b = a'/b'$. Corresponding angles of similar figures are congruent.)

I could not figure out what is expected in 7P05. The impression is one of mathematical words associated with aspects of geometry randomly strung together.

The next standard, using a different set of "mathematics words" is equally incoherent:

7P06 represent π as the ratio of the circumference of a circle to its diameter and the area of a circle to the square of its radius.

As before, I have no idea of what this means. One usually DEFINES π as the ratio of the circumference to the diameter. But what would one would mean by "represent π as the ratio ..."

One can find similar examples at all grade levels.

3. Are there specific areas that need to be updated or reworked?

I am afraid that virtually all areas need extensive reworking. If the Fordham Foundation were to review the Commissioner's Standards at this time, I believe they would receive the lowest grade, an **F**.

As an example, there is no mention of the standard algorithms. Indeed, the term "algorithm" is only mentioned twice in these standards, and this occurs only in the standards for the high school course *Advanced Quantitative Reasoning*.

The word "method" is not even mentioned through grade 7, and only occurs one time in grade 8. Here is that standard:

8A13 Determine the solutions to mathematical and real-world problems involving pairs of simultaneous linear equations (in form $y = mx + b$) using tables, graphs, and algebraic methods.

The meaning of base 10 place value notation in terms of the so called expanded form - e.g., 743 is the sum 7 hundreds, 4 tens, and 3 is not given proper emphasis. (The total discussion of expanded form is in 1N03, 2N02, 3N02, 4N04, and 5N01. But the approach there is badly flawed.) This will be discussed in more detail later in this report.

4. Are the mathematics concept/content statements grade-level appropriate? Are important concepts missing at any grade level?

Lots are missing. Indeed, too many skill level standards are missing, so the focus of the current document seems to be vocabulary and "real world" problems. For *example, there are a huge number of standards of the form* represent mathematical and real-world problems using ... *or* determine solutions to mathematical and real-world problems using .... *Indeed, the term* real-world problems *occurs over 200 times. This will be discussed further in part 3 of this report. But in too many cases, there is no previous skills development to support these standards. So I am at a loss to imagine how instruction is expected to proceed, and how students can be expected to handle anything more than the most trivial types of problems.*

For example, as has already been mentioned, one of the most dramatic issues with these standards is that there is no mention of any of the standard arithmetic algorithms. There is mention of some inefficient and non-standard algorithms, but this seems to be all. It is certainly not sufficient.

There is a very significant problem with geometry. It seems the intent of these standards is to base geometry on rigid transformations. But this is a totally different approach than the axiomatic method used since Euclid. Very few teachers will have had any experience with this approach. As a consequence, it is essential that the exposition of the standards be complete with the essential definitions such as those of congruence and similarity prominently present. But these definitions are neither presented, nor explained. For example, the following standard

GG10 identify congruent figures and their corresponding sides and angles using the definition of congruence in terms of rigid motions

makes little sense without the explicit definition that is required. In Core Standards there is a very careful exposition of all these issues, and the Texas Math Standards would do well to follow Core Standards far more closely than it does, or, perhaps better, present Geometry from the Euclidian perspective. Incidentally, the very next standard GG11 illustrates many of the other problems with these standards.

GG11 prove whether two triangles are congruent by applying the SAS, ASA, AAS or SSS triangle congruence conditions

I have no idea what it might mean to "prove whether two triangles are congruent." It would make sense to "prove two triangles are congruent by applying SAS ...." However, it is a significant challenge to prove SAS, ASA, AAS, and SSS if geometry is presented as it's done in Core Standards and imitated here, through the structure of the group of rigid transformations. So it would be important to have some detailed standards, indicating how some of these are developed. But such standards are entirely missing.

7. Should consideration be given toward adding other courses at the high school level to provide more options for students?

I would say yes. But the only courses I would suggest would be an elementary linear algebra course, and an elementary statistics course.

Flawed Standards and Other Errors

We need to understand some of the mathematical errors and imprecisions that often appear in these draft standards. Here is an example which is far from the worst.

3N18 determine a quotient using the relationship between multiplication and division (e.g., the quotient $40 \div 8$ can be found by determining what makes 40 when multiplied by 8).

First, "the relationship between multiplication and division" has never been explicitly mentioned to this point, and is not mentioned again. Are we to assume that the relationship is "the quotient $40 \div 8$ can be found by determining what makes 40 when multiplied by 8?" Then what of the quotient $40 \div 5$? Do we have to give a new explanation in this case, and indeed a separate explanation in every case?

In a similar vein, we have the standard

2N12 solve mathematical and real-world problems involving addition and subtraction within 1,000 using strategies based on place value (with and without objects and pictorial models), properties of operations, and *the relationship between addition and subtraction.*

In this standard are we to assume that "the relationship between addition and subtraction" is somehow part of "strategies based on place value and properties of operations?"

In the high achieving countries, the relationships above are crucial in instruction. The first is

> *The quotient of the number a by the non-zero number b is defined to be that number c so that $b \times c = a$ if it exists.*

Likewise, the second is

> *The difference of the whole number a and the whole number b is that whole number c so that $b + c = a$ if it exists.*

When these relationships are explicitly presented, students only have to really learn addition and multiplication well. Subtraction and division are defined in terms of addition and multiplication respectively, and thus follow easily. This pairing is a key part of instruction in the curricula of all the high achieving countries!

Here is another example representing a different kind of issue:

4N26 solve one and two-step mathematical and real-world problems involving multiplication (including scalar comparisons) and division (including interpreting remainders).

I'm afraid I have no idea what scalar comparisons could mean in this context. The term "scalar" is, as far as I know, used exclusively in linear algebra and its applications, and refers to the terms one uses to multiply vectors. It is also worth noting that the term "remainder" does not even occur till now and doesn't occur again until the fifth grade standard

5N23 determine solutions to mathematical and real-world problems involving products of positive fractions and whole numbers or positive quotients of positive unit fractions and whole numbers referring to the same whole [e.g., $1/3 \div 7$ and $7 \div (1/3)$], with fluency. (Within problems requiring division, remainders may be expressed as fractions.)

The final phrase "within problems requiring division, remainders may be expressed as fractions" represents a not so subtle misunderstanding of remainders and what they mean. In the case of whole numbers one gets a remainder when the division problem $a \div b$ does not have an answer within the set of whole numbers. For example we say $7 \div 3 = 2$ with a remainder of 1. The biggest multiple of 3 that is ≤ 7 is 6, but 3 is not a factor of 7, so we write $7 = 2 \times 3$ with a remainder of 1 to denote the truth of the equation $7 = (2 \times 3) + 1$. However, if one allows fractions, then $a \div b$ has an exact answer for every pair of fractions a and b with $b \neq 0$, so once one admits fractions into the discussion there are NO remainders any more, and the fraction referred to above is simply part of the quotient when we write the quotient as a mixed number.

Finally, what are we to make of the phrase "whole numbers referring to the same whole?"
.

4N26 and 5N23 contain the only two mentions of the term remainder until Algebra I, where, again without explaining what the term "remainder theorem" means, except in the very special case of a quadratic polynomial, we have the standard

A1Q02 apply the Remainder Theorem to a quadratic function. [For a quadratic polynomial $q(x)$ and a number a the remainder on division of $q(x)$ by $x - a$ is $q(a)$, so $q(a) = 0$ if and only if $(x - a)$ is a factor of $q(x)$.]

But in Algebra I we can assume the teacher almost certainly knows and understands the remainder theorem, so it is not really necessary to explain what is meant. So I have to wonder why it was felt necessary to tell us what the theorem is, and then only in the special case of quadratic equations.

Here is another example:

5N19 extend the definitions of, properties of, and relationship between division with whole numbers to division with unit fractions and whole numbers.

This standard mixes all kinds of things together, appears to be grammatically incorrect, and seems to me to be almost completely incomprehensible. What is the "relationship between division with whole numbers to division with unit fractions and whole numbers?" Without this being specified - and it isn't - how can we possibly make sense of this standard?

Fluency One of the key things we have learned over the last 10 years is that there are certain key skills that students must learn to use fluently or even to automaticity. In this document there are 11 standards that contain the word "fluency" and none that contain "automaticity." But it is clear that nobody has looked at 11 these standards together to see if they coordinate from grade to grade, and certainly nobody seems to have compared them to the expectations in the NCTM Focal Points, the California or Massachusetts standards, or international expectations. Here are some of these standards:

1N14 fluently produce addition and subtraction facts with sums to 10 and differences from 10 with fluency.

Aside from the strange phrasing, one should compare this first grade standard with the second grade standard

2N10 fluently produce addition and subtraction facts with sums to 20 and differences from 20.

This is the actual fluency standard. In fact, in the top standards it is an automaticity standard. If you have 2N10, there is no reason to have or require 1N14. Moreover, it is not unusual to find 2N10 as a first grade standard, but in this document it occurs in second grade.

Next we look at the initial third grade fluency standard:

3N13 solve one-step and multi-step mathematical and real-world problems involving addition and subtraction within 1,000 using strategies based on place value, properties of operations, and the relationship between addition and subtraction with fluency.

I have no idea what this standard means, and I don't understand what students are expected to handle fluently. Solving problems is such a general construct that "fluency" seems virtually impossible to attain. Indeed, I'm not even sure what it would look like.

Here is the second "fluency" third grade standard, and I'm afraid I find it almost as confusing as 3N13.

3N20 produce with fluency multiplication and division facts with products to 100 and dividends from 100.

What does it mean to "produce with fluency?"

The only fourth grade fluency standard is essentially identical with 3N13, and just as confusing:

4N18 solve mathematical and real-world problems involving positive sums and differences of positive fractions, including mixed numbers, with like denominators referring to the same whole, with fluency.

and the same issues just repeat with the first fifth grade fluency standard:

5N07 solve mathematical and real-world problems involving positive sums and differences of positive rational numbers with fluency, including decimals to the hundredths and mixed numbers.

By contrast, the next fluency standard is entirely reasonable:

5N08 determine products of up to a three-digit number and a two-digit number with fluency.

However, the key fluency or automaticity standard that prepares for this - fluently multiply two whole numbers a and b with both between 0 and 10 - or better - know the multiplication tables to 10×10 to automaticity - is not present in the document.

5N22 and 5N23 are virtually the same as the "solve problems fluently" standards that have already been discussed. Finally 6N07 and 6N08 are entirely reasonable fluency standards, but as was the case with 5N08, there is virtually no preparation for them in previous grades, so it is likely that in practice, they will only be partially taught.

It seems clear to me that the authors have not looked closely at the individual standards to try to understand what they are saying and correct them if they do not say precisely what is meant. Also, it seems to me that there has been very little attention paid to the proper sequencing of key topics as 1N14 and 2N10 and more generally the entire discussion above illustrate.

Additionally, the failure to explicitly require the automaticity standard for single digit multiplication is a huge red flag.

Real-World Problems in the Document

One or another of the two phrases **"mathematical and real-world problems"** *and* **"real-world and mathematical problems"** *occurs over 200 times in the Commissioner's Draft Texas Mathematical Standards. Putting the two topics "mathematical problems" and "real-world problems" together is, at best sloppy. The two are actually quite different and need to be handled separately. Of course, once such a discussion occurs, there would be no further need to separate them. But there is no discussion, nor any standard that differentiate between them in the Commissioner's Draft.*

Most of the time, this kind of sloppiness is not harmful. However, in the case above it really is. Below, I give a typical example of an actual real-world problem, and a pretty complete discussion of a solution for it obtained by - as it typical of actual real-world problems - first replacing the real-world problem by a precisely defined mathematical problem, solving the mathematical problem, and then interpreting the solution in the situation of the real-world problem. Mastery of each of these three steps is crucial in instruction, but they occur in totally different contexts. If the standards fail to separate them, there is no reason to assume that they will be separated in instruction.

Since the above steps are crucial for students to be able to effectively do and use mathematics, the miserable outcomes we are seeing will continue unchanged if this document is not fixed.

Here is an example of a REAL real-world problem taken from an article *How to Calculate Resolution for Publishing Photos* by Sue Chastein, its conversion of it to a mathematical problem, the solution to the mathematical problem, and finally the conversion of the mathematical solution to a series of real world actions that will achieve the desired result up to an acceptable error.

> *"Someone wants to buy a photo from me. They need it to be 300 DPI,* 5×8 *inches. The photo I have is a* $702K$, 1538×2048 *jpeg. I figure it has to be big enough! But how do I tell? The only photo program I have is Paint.NET, and I'm not sure it's telling me what I want to know. If I don't mess with it, it tells me that my resolution is 180 pixels/inch, at a size of approximately* 8×11. *If I make it 300 pixels/inch (is that the same as DPI?) I can get a print size that works, about* 5×8, *and it changes the pixel width to* 1686×2248. *Is that what I'm supposed to be doing? It doesn't seem like much of a change to the human eye.*

A lot of this confusion is because most people don't use the right terminology. They say DPI when they should be saying PPI (pixels per inch). Your photo is 1538×2048 and you need a print size of 5×8 inch, the math you need is:

pixels/inch is denoted PPI
Next we determine the horizontal and vertical PPI
that convert this photo to 5×8 inches
$1538/5 = 307.6$
$2048/8 = 256.$

That means that 256 is the maximum PPI you can get from this image to print the longest side at 8 inches without letting your software add new pixels. When your software has to add or take away pixels, it is called resampling, and it does result in a loss of quality. The more drastic the change, the more obvious the loss in quality will be. In your example it is not very much, so the loss won't very noticeable as you noted."

George Polya codified the steps involved in MATHEMATICAL problem solving as

Understand the problem
Devise a plan
Carry out the plan
look back

For real-world problems, as the above example illustrates, there are a fifth and a sixth step that are crucial. The fifth step, (actually the first step for real-world problems) is:

convert the real-world problem to a mathematical problem.

In the case above, the conversion amounted to defining the variable PPI, then determining the horizontal and vertical PPI for a 5×8 print when the original resolution is 1538×2048 pixels. But there is still one further step that is very well illustrated in the example above. One needs to

look back again

and interpret the answer to the associated mathematical problem in the original real-world situation. This also involves a number of steps and decisions, but in the context of the real-world situation, rather than the mathematics.

The Key Concepts Involved in Place-Value

Base 10 place value notation is a short-hand for writing ordinary numbers. Thus 358 MEANS the SUM of 3 copies of 100 added to 5 copies of 10 added to 8 copies of 1. Similarly 6438 means the SUM of 6 copies of 1000 added to 4 copies of 100 added to 3 copies of 10 added to 8 copies of 1. Using a further short-hand notation we write $a_1a_2a_3\cdots a_m$ with each a_i exactly one element in the set $(0,1,2,3,4,5,6,7,8,9)$ as short-hand for the SUM

$$a_1 \times 10^{n-1} + a_2 \times 10^{n-2} + \cdots a_n \times 1.$$

The sum above is called *the expanded form* of $a_1a_2\cdots a_n$, and the key part of instruction on place-value SHOULD be to have students automatically convert the short-hand notation to the expanded form when operating with whole numbers written in bas 10 place value notation.

The understanding above is crucial if students are to actually understand any multiplication algorithm for two numbers written in base 10 place-value short-hand notation. Indeed, due to the distributive, commutative and associative properties, we have

$$(a_1a_2\cdots a_n) \times (b_1b_2\cdots b_m) = \sum_1^n a_i \sum_1^m b_j 10^{i+j-2} = \sum_1^n a_i \left(\sum_1^m b_j 10^{j-1}\right) 10^{i-1}$$

and since multiplication by 10^s simply puts s zeros on the right in the number written in base-10 place-value notation, the last sum above really only involves 9 products $v \times b_1b_2\cdots b_m$ where v lies between 1 and 9. This IS THE STANDARD STAIR-STEP algorithm for multiplication, and the above is a quick description of the key steps that a teacher who understands to mathematics will follow in instruction.[1]

Later this is extended to decimals. $a_1\cdots a_n.b_1\cdots b_m$ is short-hand for the expanded form of $a_1a_2\cdots a_n$ SUMMED with

$$b_1 \times \frac{1}{10} + b_2 \times \frac{1}{100} + b_3 \times \frac{1}{1000} + \cdots + b_m \times \frac{1}{10^m}$$

It is important to note that, as a consequence, students cannot really handle decimals until they have some command of *fractions*, since decimals are defined in terms of fractions.

Finally, these ideas are extended to division. Speaking conceptually, division *is defined* as follows: the quotient $a \div b$ is defined as that number c so that $c \times b = a$. If we use the standard algorithm to write the product $c \times b$, we get a series of terms of the form $c_i \times (b_1b_2\cdots b_m) \times 10^k$, $i = 1,\ldots,n$ which are added together to get a. Thus we can try to

[1] Of course, the main problem we face is that very few lower grade teachers are trained sufficiently well in these aspects of mathematics. Consequently, they often have real problems trying to convey material they don't really understand as more than as formal sequences of operations. Of course, this problem becomes even greater when they have to teach long division.

reverse the process, first determining the largest power of 10, 10^{k_1}, which, times b is $\leq a$. then, look successively at $2 \times 10^{k_1}, 3 \times 10^{k_1}, \ldots, 9 \times 10^{k_1}$. There will be a last number in this list that is still $\leq a$. This number will be the largest digit c_1 in c! And so on. Once more, a teacher who understands what is happening *and why* can easily devise a suitable instruction sequence to teach this algorithm.

Critical skills needed to work with place-value notation In order for students to take advantage of all the power that base 10 place-value and the short-hand notation gives they need to learn some basic facts to the level of automaticity, which is even stronger than fluency. These essential facts are (1) the sum tables for all whole numbers A and B, with $A \leq 10$ and $B \leq 10$ as well and (2) the multiplication tables for all whole number A and B in this same range.

It is worth noting that these two requirements are simply not present in these standards. From the perspective of successful student outcomes alone, this lack disqualifies the document.