Comments on "The Commissioner's Draft of Texas Mathematics Standards"

Richard Askey

Before getting to detailed comments on these standards, I need to start with a story. At the annual meeting of the National Council of Supervisors of Mathematics in April, I was talking with a man and it turned out he worked for a publishing firm which had published a second year algebra book a granddaughter of ours had used. I had looked at it, and was shocked that there was no derivation of the formula for the sum of a geometric series. I asked him why this was omitted. He said that the contents of their books were determined by what states asked in their Standards. I said I doubted there was any state which wrote that students should not learn a derivation of the sum of a geometric series, but it is likely that many states had not made it clear that students should know how to derive this important result.

In the Preface, page 4, there is the statement that "many careers now require more and different mathematics, including an emphasis on mathematical reasoning". Also, on page 107, there is the following: "The placement of content in these standards is aligned to recent research, the report of the National Mathematics Advisory Panel and to the practices of countries that are high performing in mathematics." One of the recommendations from the National Mathematics Advisory Panel is that their list of topics in "Table 1: The Major Topics of School Algebra" should be covered and included in grade-level assessments by Grade 11 at the latest, which for the draft Standards would be by Algebra 2. Here is part of what it written in the more detailed description in a Task Force Report from the National Mathematics Advisory Panel.

"One of the most important identities in introductory Algebra is

$$(x - y)(x^n + x^{(n-1)}y + \ldots + y^n) = x^{n+1} - y^{n+1}$$

for all numbers *x and y."*

When x=1 this gives the sum of a finite geometric series when y is not 1. The comments continue with: "The summation formula is important in mathematics and in both the natural and social sciences. The fact that it is an elementary result that can be taught at the beginning of school algebra is not generally recognized. It should be taught early because, when it is relegated to the end of Algebra II, as is done in the standard curriculum, it does not receive enough attention and is often omitted. The result is that many students go to college missing a critical piece of information."

What is done in the current draft with geometric series? They are first mentioned in Pre-Calculus, as are arithmetic series. The relevant standard is:

PA06 The student is expected to calculate the n^{th} term, n^{th} partial sum, and the sum of a geometric series when this sum exists. [The last refers to the limiting case when n goes to infinity]

The timing for students learning how to sum a finite geometric series is too late, and it is far from clear from the standard if students will ever see how to derive the formula for the sum of a finite geometric series. Recall the story of the textbook which just stated the formula and there was no derivation in the book nor a problem in which students would be expected to derive it. The Standards have to be stated in such a way that will insure that students will have had a chance to learn how to derive such results even if the only assessment items could be done by using a formula which has been memorized.

There should be a connection between geometric series and repeating decimals, but repeating decimals are not mentioned in this draft. Irrational numbers are first mentioned in Grade 8, but there is no indication what students are expected to know about them. There are two reasonable ways to show students that irrational numbers exist. One is to give an informal argument that the square root of 2 is not rational. This is done in a Japanese eighth grade book which is available in an English translation, and is occasionally done in some U.S. textbooks. The other way is to informally show that the decimal expansion of a rational number is a decimal which eventually repeats and then find a decimal expansion which does not eventually repeat. The converse, that a decimal which eventually repeats can be changed into a rational number, i.e. a ratio of two integers with a nonzero denominator, ties up with the method used to sum a finite and later an infinite geometric series. Texas seems to want students to be told that irrational numbers exist without students seeing a mathematical reason that irrational numbers exist. This is not a way to promote the idea that mathematics involves reasoning. On page 107, in the second item in Frequently Asked Questions, there is the following comment:

Vertical development is clear as is the long-term intent of the vertical development.

- For example, we can trace number concepts, fractions and decimals into rational numbers and irrational numbers.

Since nothing is done about irrational numbers beyond saying that they can be approximated by rational numbers and put on the number line, this is a poor example to give of development, vertical or otherwise.

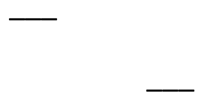
1. Is a complete and logical development of mathematics concepts followed for each grade level or course?

No, it is not. There are many instances which could be mentioned. Measurement is a topic in which U.S. students traditionally do poorly. Let us consider how area and volume are done. Area of a rectangle with whole number sides is introduced in Grade 3, 3G03. Then in 3G04 areas of composite figures made up of two rectangles are mentioned as something students should be able to compute. In Grade 4, 4A03, students are to determine a formula for the area of a rectangle, with whole number sides according to 4A04. 5N16 has multiplication of a positive fraction and a whole number represented by models, including area models. 5N18 deals with division of a unit fraction by a whole number and of a whole number by a unit fraction using objects and pictorial models, including area models. The best way to think about division, say a/b, is that a/b=c if b*c=a. Thus I see no reason to mention area models for division. Recall that area of rectangles has only been defined when the sides are whole numbers, so a bit more should be written about how an area model can be used to represent multiplication of a fraction with a whole number. Formulas for volumes of rectangular prisms are to be determined in 5A06. One assumes this has the restriction that the edge lengths are whole numbers, but this is not specified. In Grade 6, GN04 has an area model being used to present fraction and decimal multiplication and division, including the multiplication or division of a fraction and a decimal. Again, nothing has been written about a formula for the area of a rectangle when the side lengths are not both whole numbers. Since decimals are just another way of writing some fractions, there has to be something written about extending the formula for the area of a rectangle to the case when the lengths of the sides are fractions. In 6N04, I do not understand why this should be mentioned as a standard and then illustrated by three examples: 1.3*2.5, 1 3/10 * 2 ½, and 1.3*2 ½. In answer to the question of drawing area models for each and explaining why these multiplications yield equivalent products, I would draw the same picture for each and say the products are the same since 1 3/10 = 1.3 and 2 ½ = 2.5 are just different ways of writing the same numbers. If that is what is wanted, why not ask about it? If something else is wanted, I do not know what it is.

Later in Grade 6, there are two Standards, 6A10 and 6A11, which deal with area. The first is:

"illustrate and explain the relationships for areas of parallelograms, trapezoids, and triangles by decomposing and rearranging parts of these figures. For example, [a] parallelogram can be decomposed into a trapezoid and a right triangle with congruent heights; the triangle can be moved so that a rectangle is created having the same base length and height as the original parallelogram. (Figures include oblique triangles and parallelograms.)"

I have no idea what an oblique parallelogram is, so it should be dropped The argument in the example is incomplete. Consider a parallelogram of the following type. I will just draw the top and bottom sides and let the reader connect the end points to complete the parallelogram.



Explain to me how you can treat the lower side as the base and can cut off a right triangle to get a trapezoid and a right triangle and then move the right triangle to get a rectangle. Don't say to consider one of the sides which was not drawn as the base for if you do the formula you obtain only has been shown to hold for some parallelograms. There are ways of treating this case, and a teacher might decide that sixth grade is too early to do the general case, but at least the teacher needs to know that the usual textbook argument, which is what is contained in 6A10, is incomplete, and it would be useful to explain how to complete the argument, or give a reference to where it has been written. If the Texas Standards are written without requiring this case in Grade 6, it should be mentioned that a general argument will be given in high school, and then add it to the geometry course.

Finally, in 6A11 there is a comment that the formulas for area and volume should be applicable when the side or edge lengths are rational numbers. Arguments need to be made for this for rectangles and rectangular prisms, with more details for rectangles. Students should not get the idea that any formula they know in one setting can be used in a more general setting without knowing something about why this is true, when it is.

There is a serious omission when dealing with areas of rectangles. The distributive property, a(b+c)=ab+ac, should be mentioned before the first algebra course. It has been used regularly in multiplication of whole numbers and it follows easily from the area formula for a rectangle, putting together two rectangles with a common side of length *a* and other sides *b* and *c*. The commutative property of multiplication comes from the formula for the area of a rectangle when it is rotated by 90 degrees. Standard 3N15 contains these results for whole numbers, but without any reasons why they work.

The problems dealing with measurement up to Grade 7 can be fixed. Starting in Grade 7 the problems become harder to fix.

7P05 generalize the critical attributes of similarity, including invariant and covariant relationships. (If *a*, *a*', and *b*, *b*' are side lengths of two pairs of corresponding sides, then *a*/*a*'=*b*/*b*' and *a*/*b*=*a*'/*b*'. Corresponding angles of similar figures are congruent.)

What is meant by “similarity”? In the first sentence, “invariant” and “covariant” should not be used. What teacher or textbook author will know enough to be able to explain what the words mean? I do not know and have never seen them used in this context. The fact that *a* and *b* are side lengths of **triangles** is not mentioned. Depending on the definition of similarity, the last sentence is either something which needs a proof or is part of the definition of similarity. The same is true of the earlier sentence. Are these two sentences the generalizations or are they the critical attributes of similarity? If the later, what generalizations are being considered?

7P06 represent π as the ratio of the circumference of a circle to its diameter and the area of a circle to the square of its radius.

One cannot give two different definitions of the same number, which is what this standard asks for. One can be taken as the definition, and the other shown (informally) to follow. The usual way is to define π as the ratio of the circumference of a circle to its diameter, after an informal argument that this ratio is a constant, and then cut the circle into congruent wedges and make an almost rectangle from them. Then the formula for the area of a rectangle gives the second ratio. However, there is a gap in this argument which should be acknowledged. The formula for the area of a rectangle has only been shown to hold when the sides are rational, and that is not the case here. Students should be told that formulas like the formula for the area of a rectangle which have been shown to hold for [positive] rational numbers continue to hold for all [positive] numbers on the number line. No everything taught in a mathematics course can be done carefully. Some things are too hard to do when they are needed. The right thing to do then is to acknowledge that the mathematics needed to show something is too complicated to do at this time, but the gap needs to be acknowledged.

Standards7A02, 7A03, 8A02, 8A03, and 8A04 all start with “Illustrate and explain”. I have no idea what is meant by the phrase “illustrate and explain”. For the seventh grade standards, it is possible to give the start of a mathematical argument which explains why there is a factor of 1/3 in the formula for the volume of a pyramid. For a square pyramid, one can take a cube and draw lines from the center to the vertices giving six congruent square pyramids. If the edges of the cube have length a, then the volume of each pyramid is $a^3/6$, the height is $a/2$ and the base has area a^2, so there is a factor of 1/3 in the formula for the volume of each of these square pyramids. This argument can and should be made into a full argument in high school, where nothing serious is done with volumes. Two other possible meanings of “illustrate and explain” are to give a non-mathematical physical argument which cannot be made into a mathematical one, say by using sand and a pyramid and prism with the same bases and heights, or one can know a statement of the formulas and be able to tell what each part of the formula is. I would like to think that for prisms and pyramids Texas wants more than either of these.

In Grade 8, there is the following standard:

"8A02 illustrate and explain the relationship between the base area, height, and volume of a cylinder verbally and symbolically".

The omission of "geometrically" suggests that all that is wanted is to be able to state the formula without any mathematical reason why it is true. That is not adequate. A cylinder can be approximated by a prism in the same way a circle can be approximated by a regular polygon. Similarly, 8A03 dealing with the volume of a [circular] cone can be illustrated by approximating a cone by a pyramid whose base is a regular polygon.

The last of these volume standards is:

"8A04 illustrate and explain the relationship between the formula for the volume of a sphere as it relates to the volume of a cone whose base radius and height are equal and are congruent to the radius of the sphere."

There are few 8^{th} grade students, and even few middle school teachers, who could give the start of a mathematical argument to help explain this relationship. To use the same introductory words, "illustrate and explain", in these standards will likely lead textbook authors to treat all of the formulas the same way. Since the ideas involved in motivating the formula for the volume of a sphere are too complicated to do at this level, it is likely that none of these formulas will be based on mathematical ideas. For all of these volume formulas, students should be able to use the formula. However, geometric reasoning should be used when possible to motivate the formulas. If this is not done, students will continue to think that it is all magic and better students will not become interested in it. That is not a good way to attract students to STEM fields.

There is another problem in 8A. Standard "8A06 represent, verify, and explain the Pythagorean theorem and its converse using models and diagrams". I assume that what is expected is something like taking a right triangle whose legs have lengths *a* and *b*, hypotenuse *c*, drawing a square with side length *a*+*b*, and putting four triangles congruent to the first in the corners. A square is in the inside of this figure, but that has to be shown. It is clear that the sides have the same lengths, but that is not enough to show that a quadrilateral is a square. One has to show there are right angles. To do this, one needs the sum of the angles in a triangle is 180 degrees, and this is not included in the draft until the geometry course, so two years in the future. It should be included and used in grade 7 or 8. Also, what does "represent, verify, and explain" mean? How is one going to derive the converse of the Pythagorean theorem using models and diagrams? You can ask for an informal derivation of the Pythagorean theorem and knowledge of its converse, and give a full treatment in a high school geometry course.

Volume and surface area are mentioned in GM03 and a little elsewhere in high school geometry. GM03 should be more specific about the types of figures being mentioned. Similar figures should be singled out here. See a more specific comment later.

Either in geometry or more likely precalculus, there should be a mathematical treatment of volumes of pyramids and spheres. This could use mathematical induction, which is missing in the current draft, or one can use similarity, slicing and the one case mentioned earlier about decomposition of a cube into six congruent square pyramids to compete this work. This is ideal precalculus material, and teaching integration is easier when students have this background.

There are serious problems with geometry. GG04 deals with constructions. The examples start with duplicating a line segment. This is the second theorem in Book 1 of Euclid. For students who are serious about learning geometry very well, this would be appropriate for self study, but not as a standard which might be assessed. For average students, there are many more important things to learn. Here is one. When starting to learn fractions, students are often asked to make fraction strips. Among other things, a student has to fold a strip into three equal lengths, and a strip of the same length into five equal parts, etc. In elementary and early middle school, this can only be done approximately. Now, with knowledge of similarity, it is possible to divide a line segment into say 7 parts each of which has the same length, and the method is general and works for *n* parts.

I strongly recommend dropping the example of duplicating a line segment. Also, drop duplicating an angle. One very important fact about lines and circles might be included in the unspecific results involving circles and lines in GC02; the radius of a circle and a line tangent to the circle are perpendicular when they have a common point on the circle; but it might not be treated if it is not mentioned explicitly.

One serious problem with geometry is that no foundational basis has been described. Is congruence “same shape and same size” or it is defined using translations, rotations and reflections? Is similarity defined in terms of congruence or is congruence defined as a special case of similarity as in the previous Texas Standards? If you are serious about school mathematics, there needs to be a solid foundation and “same shape and same size” does not provide this. Here is what seems to be written about congruence.

8G01 The student is expected to generalize the properties of orientation and congruence of rotations, reflections, and translations of two-dimensional figures on a coordinate plane. For example, rotations, reflections, and translations preserve congruence of two-dimensional figures.

What does this mean? If congruence is defined by two figures being congruent if one can be shown to be the same as the other by being moved by a sequence of translations, rotations, or reflections, then the last is just the definition. If it has some other meaning, what is the definition of congruence?

8G02 …differentiate between transformations that preserve congruence and those that do not.

Again, what does this mean? Is this just vocabulary: translations, rotations and reflections preserve congruence and other transformations do not, or is it something else? What about the transformation (x,y)-→(3x/5+4y/5,-4x/5+3y/5). Does this preserve congruence? I think this is too hard a question for Grade 8, but an appropriate question after trigonometric functions have been studied and the addition formulas have been learned. I could not find anything related to this in precalculus, and the closest I could find is AQRG03.

GA03 How is one to prove the condition for perpendicular lines in terms of slope using algebraic methods? Perpendicularity is a geometric condition and there has to be a geometric basis for treating it. This may use the Pythagorean theorem. While use of the Pythagorean theorem frequently is algebraic, it is really a theorem in geometry.

2. Have the correct vocabulary and terminology been used? Where can changes be made for accuracy and clarity?

Often correct words are used, but some serious errors have been made.

GS03 and GS04 both contain the words "choosing from various formats of proof such as paragraph, flow, two-column, coordinate, or transformational". Here two different things are grouped together. The words "paragraph", "two-column" and "flow" [which is short for "flow chart"], are different ways of writing the steps in a proof, but "coordinate" and "transformational" refer to different ideas in a proof. The same error occurs in GG13, 14, 15.

Another example is GG01 "distinguish between undefined terms, definitions, postulates and theorems using mathematical induction and deductive reasoning". Teachers and students both need to know the difference between a definition and a theorem. They also need to know how to use definitions, and so know that standard 7P06 is inappropriate since it in effect gives two different definitions of π. This is a good example to elaborate on, since π could be defined as the ratio of the circumference of a circle divided by its diameter, and then the fact that the area of a circle is π times the square of the radius is a theorem, or π could be defined as the ratio of the area of a circle divided by the square of the radius and then the connection between the circumference and the diameter becomes a theorem. Another example is even and odd numbers. These occur in 2N14. No definition is given. At the second grade level there are two possible ways of defining even numbers. One could say the number of objects in a set is even if the set can be split into two groups each having the same number of objects. Another way to define even would be to say that the objects can be taken out two at a time and nothing is left over. Later, students will use a third property of even numbers. A whole number is even if it ends in 0, 2, 4, 6, or 8. Only one of these properties can be taken as a definition, the others have to be shown to be equivalent to the definition. Even this is not the full story, since one sometimes wants to expand the class of objects and the original definition might not make sense in the more general case. For example, 0 is considered as an even number, and so is -2, etc.

There are more serious problems with GG01 which can be solved by dropping a few words. Drop the last six words. I have no idea where the words "mathematical induction" came from in this setting. Mathematical induction should be included in the precalculus course. Its use here will make the Texas standards a laughing stock. Deductive reasoning is not as bad, but it is not really appropriate either. This is written with the assumption that the last part "using mathematical induction and deductive reasoning" describes methods of distinguishing between the listed words. Neither of these is what is needed to do that. This could also Abe read as "mathematical induction and deductive reasoning" being various ideas used in proofs. If so, this should be a separate standard and various forms of argument should be listed such as enumerating cases, proof by contradiction, and others.

About undefined terms, consider the following two settings for high school geometry. One is a synthetic approach without the use of coordinates. Then "point" is treated as an undefined object which has various properties, one being that given two points, there is one line containing both points. In a coordinate approach in two dimensions, a point is a number pair such as (3,1). I am not sure you want to get into a discussion of "undefined terms". I do not know why the word "postulate" is being used. A postulate is an assumption which has been made. I think it would be better to use the word "assumption". My advice it to keep this simple and just include definitions, assumptions and theorems.

5N23 This ends with the comment: "Within problems requiring division, remainders may be expressed as fractions." Division is an inverse of multiplication as was mentioned above. When dealing with division in which the number system has not been extended to make arbitrary divisions possible, one does not write 5/2 = 2 R1, but 5 = 2*2 + 1. There are good reasons for requiring this. Students need to learn the properties of "=", and one of them is that if A=B and C=B then A=C. Is this true if 5/2 = 2 R 1 and 9/4 = 2 R1? No, since 5/2 is not equal to 9/4. Once fractions have been introduced, then one can change 5 = 2*2 + 1 to 5/2 = 2 + ½ and this is frequently written as 5/2 = 2 ½. That is the essence of 5N23, but more needs to be written to tell textbook writers about a common error in textbooks. Also, the phrase "Within problems requiring division, remainders may be expressed as fractions" is not really correct. Once one has fractions, then the division of 5 by 2 is exact and there is no "remainder". There may be a fractional part which is not zero, and students need to know how to use this when dealing with some problems.

One serious problem in the use of words is the almost complete absence of definitions. For example, P705 uses the word "similarity", It is a focal topic in grade 7, but not defined or described. Many of our school textbooks mention that congruence is a special case of similarity but do not have definitions which can be used mathematically until a few years later if then. Teachers need to know that congruence comes before similarity and similarity is defined in term of congruence.

3. Are there specific areas that need to be updated or reworked?

Kindergarten

KN01 The emphasis here is vocabulary rather than understanding what is being done. One can make a case for counting to 100, but 40 should be adequate for kindergarten.

KN02 When counting objects in a set, students need to know that the last number they say is the number of objects in the set.

KN05 I would restrict this to sets with at most 20 objects.

KN13 Before doing subtraction within 20, students should learn how to count backwards from 20 and smaller numbers.

Grade 1

After 1N01 add a standard containing counting backwards from numbers up to 100, both by 1s and 10s.

1N10 There should start to be some simple two step problems. John has 5 books and Joan has 7. If John gets 4 more books, how many books do John and Joan have together, or how many more books does John now have than Joan has?

What is the difference between KN12 and 1N15? Both limit the size of the problems to 20. There should be some progression between grades. Singapore Primary Mathematics has problems like 53 – 6 done with a drawing and decomposition of 53 to 40 and 10 and 3, with the six taken from the 10 to leave a picture of 40 + 4 + 3. They also start multiplication and division with small numbers and pictures. Multiplication can wait, but you need a little more on addition and subtraction.

Grade 2

A number line should be introduced and how addition and subtraction of whole numbers act on a number line should be a standard. The number line is mentioned in 2M05 and 06 but the connection with addition and subtraction is not mentioned.

2N06 The figure here is neither a regular polygon nor a strip diagram.

2N08 The phrase “like denominators” is ambiguous and should be replaced with “equal denominators” when that is what is meant.

2N09 Drop the first “and”.

2N14 40 should be replaced by 20 since splitting a set of more than 20 objects takes more time than it is worth. What definition will be used for odd and even?

2A02 the word “multiplication” should have been introduced in 2N15 or 16. Do you want to include problems with 3 times 8 4 times 6, and 2 times 12?

2A01 add “and solve” after “represent”.

2G03 what you want here is for students to recognize these figures. How is a student to classify a cone or a sphere based on properties like those suggested?

2G05 Are there to be pictures with this standard? If not, more specification needs to be given.

2M04 Add: find the difference between two times given to an hour, including some when one time is AM and the other is PM.

Grade 3

The first focal area is multi-step addition and subtraction problems with whole numbers within 1000. Based on released NAEP items, it is essential that students have some practice with addition of four two digit numbers where regrouping is needed. Also, fluency seems only to be mentioned in 3N13, and there it seems almost to be on oversight. Students need more than this level of ability in computation.

Reorder 3N07, 08, 09 to start with 08, then 09, and then 07.

3N20 You do not mean facts with products to 100, since 7 times 13 is not basic, but the multiplication facts up to 10 by 10 and the corresponding division results are fundamental.

3N15 No emphasis is put on these very important facts. What does 2 * 3 * 4 mean? By the usual convention it is (2*3)*4. You need to also have (2*3)*4=2*(3*4).

The associative and commutative properties for addition should be mentioned in Grade 2; probably not the name, but the property, and commutativity in first grade.

3A01 should also have “and solve” after “represent”.

Earlier, in 3G01, what properties of polygons with more sides than 4 do you want students to point out specific properties? In school mathematics, there are three reasonable conditions which could be given, regular, convex or not convex, and cyclic. For a polygon to be regular it has to have sides of the same length and angles with the same measurement. Angle measurement comes in Grade 4, so this property will have to wait. Convex and not convex can be introduced when the angle sum of a triangle has been introduced and you want to extend it to polygons. In the current draft, that is high school geometry. Cyclic is high school geometry. Rewrite 3G01 to refer to quadrilaterals.

You should have a standard saying that students should be able to fold a torn out sheet of paper to get a right angle. There should also be a comment that triangles are rigid but quadrilaterals are not.

3G03 This standard can be used to show informally why the distributive property is true. Add it either here or in grade 4.

Grade 4

4N21, 22 Represent the product… using arrays, area models or equations. What is being asked here? Is this a way of saying that students should be able to compute using arrays or area models, or just draw a picture of what multiplication means? I have no idea what is meant by using equations to do this. Would 2345*7=16415 give an equation of the sort you ask for? I suspect not.

4N17 and 18 “like” denominators again. This should be “equal”.

4N26 What is “scalar comparisons”? Delete it.

NA03 determine the formulas for the perimeter of a rectangle, including the special form for the perimeter of a square, and the area of a rectangle.

Perimeter is the sum of the lengths of the sides of a polygon. Change the first part to "be able to change the formula for the perimeter which comes from the definition, $a+b+a+b$ to $2a+2b$, and to $4a$ when the rectangle is a square, where a and b are the lengths of the sides of the rectangle. About the area of a rectangle, students should be able to explain why the area of the rectangle above is $a \times b$ when the lengths of the sides are positive whole numbers.

Grade 5

5N06 What is the point of stressing addition of 1/5 + 0.3. The essential point is that 0.3 is just another way of writing 3/10, and that is what students need to know.

5A05 This is the first mention of exponents. What are students expected to know about exponents in fifth grade? In 5A06, volume of a cube is written as $s \times s \times s$ *rather that as* s^3.

Grade 8

8N01 Change "the rational…" to "a rational…".

8P07 and 8P11 A very important non-proportional relation is the area of a square to the length of the side. Let me suggest adding "linear" to get "non-proportional linear situations".

8A09 Drop the first "or".

8A12 Since this is the first standard which mentions having the variable on both sides of the equal sign, it would be a good idea to start with integer coefficients, so change to "should include examples with integer coefficients and constants and also with rational coefficients and constants".

There should be a standard about knowing why the graph of a linear equation is a straight line and why an equation for a straight line plotted in the plane is a linear equation. This could be done in Grade 8 or in Algebra 1. If in Algebra 1, it needs to be before A1L05 which assumes this fact is known.

Algebra I

A1Q07 Is there to be a derivation of the quadratic formula? Recall the story this review started with, where no derivation of something important was given because a derivation was not mentioned explicitly in many state standards.

A1O02 determine the meaning of a and b in an exponential function of the form $f(x)=ab^x$ in mathematical and real world problems.

I hope this is not all you want. Sometimes a and b will have a meaning in a real world problem, but in general f(0)=a is all one can say about a, and b determines the rate of growth or decay of the function f(x) when x becomes large. These two facts should be known even if there is no other meaning to a and b.

A1A13 What is meant by factored form for a polynomial of degree one? Is $2x + 5 = 2(x + 5/2)$ what you have in mind? How is this different than $5(2x/5 + 1)$? My advice is to drop first degree polynomials.

A1A14 Drop "one or".

A1A15 Do you mean that when f(x)=0 when x=1 and x=2, then f(x)=k(x-1)(x-2) for a constant k? I suspect you want to assume a bit more than the roots. About the graph, you have to have three points on the graph to find the function. Once you know three points, the graph is not needed. Rewrite to get at the essence of what you want students to be able to do.

Geometry

GA02 GA03 belongs before GA02. You can not do GA03 without some geometry as was mentioned earlier. Just stating the slope condition for perpendicular lines is not adequate, although that is what is done in many textbooks.

GA04 There should be a reasonable progression through the years. Algebra 1 should have a treatment of the connection between the graph of a linear function and a straight line. Geometry should include the connection between the graph of a circle and an equation which represents the graph. A parabola would fit into Algebra II more naturally than here. Then the sequence contains ellipses and hyperbolas in Pre-Calculus.

GG06 Move this much later. What little is done with spherical geometry should come late, after a lot has been done with Euclidean geometry.

GG14 Add to the statement about the medians that they meet at a point which is 2/3 of the way from each vertex to the opposite side. [This is clear from a synthetic proof, and of a proof using general coordinates for the vertices, but is hidden when the coordinates are placed at (0,0), (a,0), (b,c).]

GS04 Drop the comments on the type of proof.

GM03 This standard is stated in too general a form. You want to deal with the effects of dilation on similar figures for length and area in two and three dimensions, and volume in three dimensions.

Algebra II

A2F01 A base should be included when using log, as $\log_b$. This occurs twice.

A2Q07 Include directrix as a given. I would separate the conditions, putting focus and directrix as the first way to get a parabola, and mention some of the others as secondary possible conditions.

A2Q10 Solving cube root equations for complex solutions makes more sense in Pre-Calculus after the appropriate trigonometry has been done.

Properties of exponential and logarithmic functions What is the essential property of e^x? It is the addition formula $e^{x+y} = e^x e^y$. The same formula holds for other bases. I have tried to find this for at least one base and have failed. This and the corresponding formula for logarithms have to be added. For both logs and exponentials there is a second formula which should be added. For logs it is $\log a^x = x \log a$. What is stated in PF07 is not adequate.

A2A06 Drop this. It can be replaced by: be able to use the following: If f(x) is a polynomial, then f(a)=0 if and only if f(x)=(x-a)g(x) where g(x) is a polynomial.

Pre-Calculus

Either in Algebra II or Pre-Calculus there should be a standard on the connection between the coefficients of a quadratic function and the zeros of this function. This has not been done in the U.S. for a moderately long time, but it is important and is done in East Asian countries. This can be phrased for equations rather than functions, but either way would be fine. In Pre-Calculus there should be a problem like the following, which also relates to quadratic equations.

Find the intersection points of the unit circle and a line with slope t which contains the point (-1,0). The standard could be "solve quadratic equations where the coefficients contain specific parameters. Then give the problem above as an example. Pre-Calculus will not be taken by all students so it is time to give some more serious problems. This one gives a rational parameterization of the unit circle and with a simple change of variables gives Pythagorean triples.

PF04 As a second part, include the following. If f(x) is defined for $-a < x < a$, use the definitions of even and odd functions to show that f(x) = e(x) + o(x) where e(x) is an even function and o(x) is an odd function.

PG12 Why ask for students to know how to find the eccentricity of a hyperbola? For elementary applications, the eccentricity of an ellipse is important, but I do not know corresponding uses of the eccentricity of a hyperbola.

PG13 How is a student going to "determine the conic section formed when a plane intersects a double napped cone"? I would drop this, or change "determine" to a weaker word.

PG14 At least the addition formulas for sine and cosine are mentioned, but only to "use", not be able to derive and use. The difference formula for tangent allows one to find the angle between two lines. This is a nice application and it suggests that the addition formula for tangent should be included.

PM05 and 06 Students should be asked to derive and use these formulas.

PA02 This is a poor way of writing a standard. What series will be expected to be summed? An arithmetic series, the series in the binomial theorem, and finite and convergent infinite geometric series and the only ones, and each of these is covered by a different standard. This could be rewritten as follows: be able to write out the terms of a series when it is given in sigma notation.

PA07 Polar coordinates need to be a separate standard.

PA09 determine powers and all the n^{th} roots of complex numbers.

Let us do this for 2+3i. First one has to find the modulus of this number, which is related to polar coordinates. The only place where polar coordinates are mentioned is in PA07, which fortunately mentions complex numbers. Then one needs to be able to multiply complex numbers in trigonometric form. That is also a standard. If one is interested in the n^{th} root, then one needs an old formula found by de Moivre which needs mathematical induction to prove it. That step is missing in this draft. How is a student going to do this problem? There is a related problem which uses the addition formula for sine. Write $2\sin(x)+3\cos(x)$ as $a \sin(x+t)$. A physics teacher will be very glad if students have learned this, and only much later in physics will it be useful to be able to find all n^{th} roots of a complex number. Put it in as a standard and either drop finding all of roots of a complex number or restrict it to square roots and cube roots.

PA11 and 12 This is incoherent. Pascal's Relation comes from $(a+b)^{n+1}=(a+b)(a+b)^n$. Nothing has been written about how binomial coefficients are to be defined. They could be defined combinatorically, but then there is a problem of identifying them with the coefficients in the expansion of $(a+b)^n$. They could be defined as the coefficients in the expansion of this function. In neither of these ways is it reasonable to use Pascal's Relation, which has not been derived, to do something which is easier to do than it is to first prove Pascal's Relation. Also, it would be better to not only have a recurrence relation for binomial coefficients, but also to have a formula for them. This used to be done in high school and the formula has many uses.

PA13 What type of problem do you have in mind that requires changing the base of a logarithm? The change of base formula is something students should be able to derive, but it is of secondary importance, while $\log(xy)=\log(x)+\log(y)$ is of primary importance, yet it was not even mentioned in this draft.

PA17 One cannot solve polynomial equations using Descartes' Rule of Signs. This gives a bound on the number of real roots The Fundamental Theorem of Algebra tells you that a polynomial has a root, and then by a simple argument, that a polynomial of degree n has n roots. I have no idea how one would use the Fundamental Theorem of Algebra to solve a polynomial equation, and I bet the authors of this draft do not know how to do this either. Rewrite this. You might mention that some equations have to have their solutions given approximately.

PA18 What are "critical numbers"? What you mean are solutions of the corresponding equations. Write this rather than jargon.

Advanced Quantitative Reasoning

I do not have any experience teaching a course like this.

AQRN05 How are students going to get data which is not easily measured so that they can then analyze it? If it is given to them, for them it is easily measured. What does this standard mean?

I have a colleague who tried to develop a good quantitative reasoning course for college students. After about five years he began to think it was impossible to develop one which required students to think and they could do an adequate job of mathematical thinking. He was a careful worker and if he could not develop something which satisfied him, I have doubts that the course Advanced Quantitative Reasoning will get many students to the level where they can "demonstrate reasoning skills in developing, explaining, and justifying sound mathematical arguments, and analyze the soundness of mathematical arguments of others", which is AQRN02. What textbook does the committee have in mind for this course? Teachers cannot teach it without a good textbook, and I do not know any.

Mathematical Models with Applications

There may be textbooks for a course like this which could be used to map out a decent course. However, one would have to be very careful for there are books like "Earth Algebra" which some might think would be appropriate for a course in modeling. It has little substance mathematically or in other ways.

MMAD03 One will almost surely find that graphical ways of recording data are not the major sources of error.

4 Are the mathematics concept/content statements grade-level appropriate? Are important concepts missing at any grade level?

3N01 includes the notion of bundling. This should have been introduced in grade 2 and in some sense, in first grade.

2N15 and 2N16 This is first grade material in Singapore Primary Math and the notation of multiplication is used in the Singapore books. Why not use the notation here?

Many other examples were given in answers to questions 1, 2, and 3.

5. Are the Student Expectations (SEs) clear and specific?

Sometimes, but a number of cases where this is not true have been mentioned above.

6. Is the subject area aligned horizontally and vertically?

Sometimes, but some and almost surely not all examples when this is not true have been mentioned above. Every time I read this draft I find more problems.

7. Should consideration be given toward adding other courses at the high school level to provide more options for students?

A course on data and statistics, but make it not recommended for students who will use statistics in college. They should learn this material in college, and not, to quote a statistician friend, have to relearn the inadequate treatment she has had to deal with in the major research university where she teaches.

8. Do you have any other suggestions for ways in which the mathematics standards can be improved?

A serious rewriting needs to be done. The draft starts out in Kindergarten asking for much more than is done in East Asian countries, and by second grade has started to fall far below what is taught and learned in Japan, Singapore and Hong Kong, and most likely in China.

There is no provision for getting say 25% of high school students through a first course in calculus. There are many mathematicians who wish there was only a small calculus program in high school, or even no calculus courses, since they wish that a better algebra and geometry program would be done instead. The high school program in this draft is not what these mathematicians would like, and there is no provision for getting students to calculus say in their last year of high school. A few suggestions for added material have been given, but more changes would have to be made in algebra and geometry as well as pre-calculus.

The only separation comes in the last year of high school. Do you know any country which tries to teach the same mathematical level to all students through grade 11? I do not. What do you propose doing about this? Recall that on page 107 there is the following statement:

"The placement of content in these standards is aligned to recent research, the report of the National Mathematics Advisory Panel and to the practices of countries that are high performing in mathematics."