Review of the Commissioner's Draft of the Texas Mathematics Standards

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In my review I shall compare the Commissioner's Draft (or "Draft" for short) to the Common Core Standards for Mathematics (or "CCSM" for short). I shall also examine how it measures up to the NCTM Focal Points ("Focal Points") and to the recommendations of the National Mathematics Advisory Panel ("NMP"). My comparisons to the CCSM should not be construed as advocacy of the Common Core – to the contrary, I consider it a mistake that California, Indiana, Massachusetts, and Minnesota have replaced their own mathematics standards with the CCSM.

Summary of the main conclusions

The authors of the Draft scrupulously avoid pedagogical prescriptions, which are standard fare in the CCSM and most state frameworks. I applaud this decision to leave pedagogical choices to local schools and districts. Unfortunately the Draft falls short of the CCSM in other ways. It often aims lower than the CCSM – it is *less rigorous* than the CCSM, certainly not more rigorous as is claimed in the FAQ section. In quite a few cases the Draft uses imprecise, awkward, or misleading language.

Specific comments

My comments are organized around three types of problems in the Draft: omitted or underemphasized topics, flaws in the development of certain ideas, and mathematically inaccurate, ambiguous, or misleading language.

1) Missing or underemphasized topics

Most glaringly, the standard algorithms of arithmetic – i.e., the standard algorithms of addition, subtraction, multiplication, and division – are completely missing; they are not even mentioned in the Draft. In contrast, the CCSM calls for fluent use of these algorithms in grades 4, 5, and 6, respectively, one to two years later than the Focal Points or the NMP recommendations. The standard algorithms constitute a major topic in all high achieving countries. I cannot fathom why the Draft is silent about the algorithms.

The CCSM and the Focal Points mention prime and composite numbers, in grades 4 ad 5, respectively. It is a mistake for the Draft to omit this topic.

Negative integers are not mentioned at all in an explicit manner. They do come up implicitly, of course, when rational numbers are introduced in grades 6 and 7. But even at those grade levels the expectations are rather unclear – that is a case of mathematical ambiguity, which I shall take up later. In any case, the earliest instance of negative numbers in the Draft appears in grade 6. That is too late. Students should encounter negative integers before negatives are introduced in the setting of rational numbers. The Focal Points, for example, call for an informal, contextual introduction to negative numbers in grade 5, which is consistent with the practice in high achieving countries. It is true that the CCSM also waits until grade 6 – a mistake, in my opinion – but at least manages to introduce negative numbers in a clearly laid out, coherent fashion. The Draft muddles this topic badly.

Money as a model for integer or decimal arithmetic is mentioned only at grade level 4 (but coins, as objects to be counted, are mentioned also in standard 1N01). In the CCSM and the better state frameworks money as a model for arithmetic is used more systematically. The authors of the Draft may argue that general references to "real objects, manipulatives, paper/pencil, and techniques such as …" include money implicitly. However, money is so obviously applicable to the practice of arithmetic, and so close to the everyday experience of many students that it deserves a more prominent role than the Draft gives it. The 7th and 8th grade standards on proportionality and ratios do not include (simple) interest, tips, and discounts as examples. As with the case of money, they are often encountered in real life. Both the Focal Points and CCSM make a point of including these examples.

2) Flaws in the development of certain ideas

Our system of writing numbers in base ten notation is fundamentally based on the idea of place value. In the Draft, the words "place value" first occur in the standards 2N12 and 3N16, in the context of "using strategies based on place value …". On the other hand, the idea of place value is clearly implicit in the standards 1N03 and 2N01. The words "place value" should first appear in those standards – CCSM introduces the notion in grade 1. Before students can *use* place value, they need to *understand* what it is. The *under-standing* of place value can be, and should be, perfectly well assessed, independently of the correct *use* of place value.

The Draft covers fractions in grades 2 through 6. That is fairly common in the stronger US state standards and in high achieving countries. But first of all, the scope of the Draft's fraction standards is sometimes unclear. The Focal Points talk of "commonly used fractions" in grade 3, and the CCSM mentions in a footnote that third graders are only expected to deal with fractions having denominators 2, 3, 4, 6, or 8. The Draft is silent on this matter. Are Texas third graders really expected to deal with arbitrary positive fractions? Secondly, the Draft contains many standards talking of "representing", "comparing", "explaining", and "decomposing" fractions. The only standard that mentions *computing* with fractions at least implicitly is 4N18: "solve mathematical and real-world problems involving positive sums and differences of positive fractions, including mixed numbers, with like denominators referring to the same whole, with fluency". What does this mean? Solving problems fluently? Solving problems involving sums and differences of fractions with like denominators? The Focal Points, the CCSM, the NMP all ask for fluent performance of the operations of addition and subtraction of fractions – not only with like denominators! – by the end of grade 5. That is also the expectation in high achieving countries. Mathematically, it is an important precursor to Algebra.The Draft should contain an unambiguous standard on the computation with fractions.

Traditionally US high school geometry, as well as its 7th and 8th grade prerequisites, have been based on Euclidean geometry. In geometry, students first encounter formal proofs, the notions of axioms, hypotheses, converse and contrapositive of a statement. Rigid transformations and dilations are covered, but not as a fundamental tool for the development of the subject. For example, the 2000 California and Massachusetts frameworks take this approach, as do the geometry standards of most high achieving countries. In contrast, the CCSM geometry standards are based on the understanding of transformations and their effects. The CCSM geometry standards are mathematically sound, but in the opinion of many critics – myself included – they are misguided; among other problems, they require substantial retraining of teachers, who are entirely comfortable with the traditional approach. Unfortunately the authors of the Draft are following the CCSM approach, but not nearly with the same degree of understanding and coherence. The geometry standards should be re-written entirely, possibly using the 2000 California or 2000 Massachusetts framework as a model. However, if the authors of the Draft *really* want to follow the CCSM approach to geometry, they need to emulate not only its direction, but also its clarity and cohesion.

3) Imprecise, awkward, or misleading language

1N04 What does it mean to "generate a number that is … equal to a given number"?

2G02 I can see no reason why this sequence consists of "quadrilaterals (including parallelograms), pentagons, and octagons". Either make it "quadrilaterals, pentagons, and hexagons", or "quadrilaterals, pentagons, hexagons, heptagons, and octagons". If the authors do not trust readers to know that parallelograms are particular instances of quadrilaterals, then many standards will have to be re-written in a more pedantic manner.

3N09 As mentioned earlier, it seems unlikely that the authors mean to include all quotients of strictly positive integers a/b in this third grade standard. I suspect, but cannot be sure, that they meant to include the restriction $a \leq b$, and also want to restrict the possible choices of b.

4N18 Reading this standard literally, one must conclude that it asks students to solve these types of problems with fluency. If so, this would be the only instance in the Draft of students being expected to *solve problems with fluency*. I rather suspect that the authors wanted to ask for fluent computation, which would be conflating two entirely different matters. The standard should say: "Solve mathematical and real-world problems involving sums and positive differences of positive fractions, including mixed numbers. Problems should only involve fractions and mixed numbers with like denominators". In addition, there should be a fifth grade standard asking for fluent addition and subtraction of fractions, without the restriction of like denominators.

4N19 Again I doubt that the standard, as written, reflects the intention of the authors. It would make sense if the second sentence said: "For example, if ½ is added to a positive number a, the sum must be greater than a and less than a+1". The second sentence, as written, is technically correct, but nonetheless silly as an example of the meaning of this standard.

8A06 The consensus of framework writers in the last twenty or thirty years is that students should know the Pythagorean theorem, be able to apply it, and to understand its proof – not to be able to reproduce its proof necessarily, but more than just understanding the statement. The CCSM says "Explain a proof of the Pythagorean theorem and its converse" and the Focal Points ask students to "explain why the Pythagorean theorem is valid". Both of these formulations are reasonable. On the other hand, I don't know what it means to "represent, verify, and explain the Pythagorean theorem". How would one *represent* the Pythagorean theorem? And the statement of the theorem is pretty straightforward, so there is not much to explain. The Draft should adopt language similar to that of CCSM or the Focal Points.

A1L01, A1L07, A1Q01, etc. These standards involve the notion of *function, domain, and range of a function*. While I regard it appropriate to talk of functions informally before Algebra I, at that level students should know the technical definition of function, for example the definition in the CCSM standard "Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range".

A1L10, A1L11, A1L7, A2L06 These standards ask to "graph the solution to … inequality", or to "solve … inequalities". The language suggests a degree of definiteness that is not appropriate. Linear equations typically – but not always – have a single solution, and quadratic equations typically have two. But the solution set of an inequality is typically infinite. For this reason, the CCSM mostly – though not entirely – avoid talking of "the solution" of an inequality; instead they talk of the "solution set", "representations of solutions", or the "constraints" imposed by a solution. Also, the standards AL10 and AL11 could easily be combined into a single standard. I have some idea what the authors have in mind when they ask students to solve "inequalities for which the application of the distributive property is necessary", but is a rather awkward description of certain types of problems. Better give an example.

A1A14, A1A18 The terms “trinomial” and “literal equation” are rather quaint by now, and are not used in college mathematics. Both the CCSM and the Focal Points avoid them. So should the Draft.

GG01 This standard makes no sense. One can “distinguish between undefined terms, definitions, postulates and theorems”, but not by “using mathematical induction” or “deductive reasoning”. Also the parallel reference to “mathematical induction” and “deductive reasoning” suggest the authors were thinking of “inductive reasoning”. Mathematical induction is something entirely different.

GG03 Better say: “Understand that a conjecture can be disproved by giving a single counterexample”.

GG04, GG05 I don’t know what it means to “represent the construction” of something. Also, “formal geometric constructions” generally refers to compass and straightedge constructions. A construction using paper folding or software should not be regarded as a “formal construction”. The corresponding CCSM standards are much clearer.

GG13, GG14, GG15, GS03, GS04 These standards conflate several very different expectations. Proving geometric statements is one thing, presenting the proof in “paragraph, flow, or two column” format is something else – probably not worth mentioning at all – and “coordinate and transformational” describes *methods of proof*, not *presentation of the proof*. Lastly, a standard asking for *proofs* should generally not also ask for *applications*. Keep those matters separate! Again, the corresponding CCSM standards are much clearer.

GM05 The “total area” of three dimensional objects has a clear meaning, but the “lateral area” depends on how the object in question is located in space. I would avoid talking of lateral area here.

A2F01 It is more common to use “maxima” as the plural of “maximum”. Also, it should be $\log_b(x)$, not $\log(x)$.

A2L03 One can express a system of linear equations in matrix notation, but one does not “replace the system by the matrix”.

A2L04 Solving a system of linear equations by using matrix notation *is* an algebraic method. Maybe the authors mean to ask students to be able to solve systems of linear equations both by using matrices, and without using matrices.

A2A01 This standard is a logical prerequisite for A2Q08 and A2Q10, and should precede the latter two.

A2E08 It is possible to give examples of "absolute value equations that have complex roots", but this is too obscure a topic to deserve inclusion in a standard.

PA06 The sum of a finite geometric series always exists. I presume the authors have infinite geometric series in mind, in which case they should say so.

PA11 The binomial theorem *is* the "expression of $(a+b)^n$ in expanded form"; one does not *use* it for this purpose. Students should know the binomial theorem and be able to use it.

PA12 These coefficients are commonly referred to as the "binomial coefficients". In effect, this standard asks students to understand the binomial theorem. It might as well say so.

PA17 The fundamental theorem of algebra is an abstract existence theorem that is rarely, if ever, used to *find* solutions of polynomial equations. The most one can ask for at the high school level is that students know the statement of this theorem.

AQRG02, AQGRG03 These standards should give some idea of the intended scope, perhaps by mentioning examples.

AQRG04 It would be better to refer to "indirect measurements" than to "inaccessible distances"; what is inaccessible is not the distance, but the object being measured.